6.3 Ecological Models (continued)

last time: predator-prey

$x' = x(a - py)$ prey

$y' = y(-b + qx)$ predator (uses prey as food)

today: competition system – two species going after the same food source but not each other (squirrels and chipmunks)

$$\frac{dx}{dt} = a_1 x - b_1 x^2 - c_1 xy = x(a_1 - b_1 x - c_1 y)$$

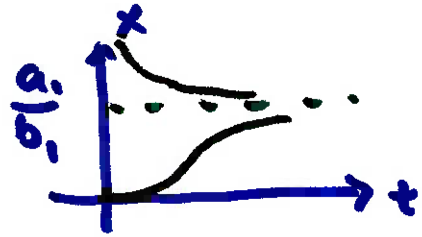
$$\frac{dy}{dt} = a_2 y - b_2 y^2 - c_2 xy = y(a_2 - b_2 y - c_2 x)$$

$a_i, b_i, c_i > 0$

if y is not present, $x' = a_1 x - b_1 x^2 = x(a_1 - b_1 x)$

logistic growth

grow until carrying capacity $x = \frac{a_1}{b_1}$



both x, y grow logistically
the presence other reduces the rate
and reduces carrying capacity

example

$$x' = x(1-x-y)$$
$$y' = y\left(\tfrac{3}{4} - y - \tfrac{1}{2}x\right)$$

cp: $(0,0)$, $(1,0)$, $(0,\frac{3}{4})$, $(\frac{1}{2},\frac{1}{2})$

$(0,0)$	$(1,0)$	$(0,\frac{3}{4})$	$(\frac{1}{2},\frac{1}{2})$
both die	y dies	x dies	coexistence

given $x(0)$, $y(0)$, what happens if $t \to \infty$

$$J(x,y) = \begin{bmatrix} 1-2x-y & -x \\ -\frac{1}{2}y & \frac{3}{4}-2y-\frac{1}{2}x \end{bmatrix}$$

$$J(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{3}{4} \end{bmatrix} \quad \lambda = 1, \tfrac{3}{4}$$ source, unstable

$$\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$J(1,0) = \begin{bmatrix} -1 & -1 \\ 0 & \frac{1}{4} \end{bmatrix} \quad \lambda = -1, \tfrac{1}{4} \qquad \text{saddle, unstable}$$

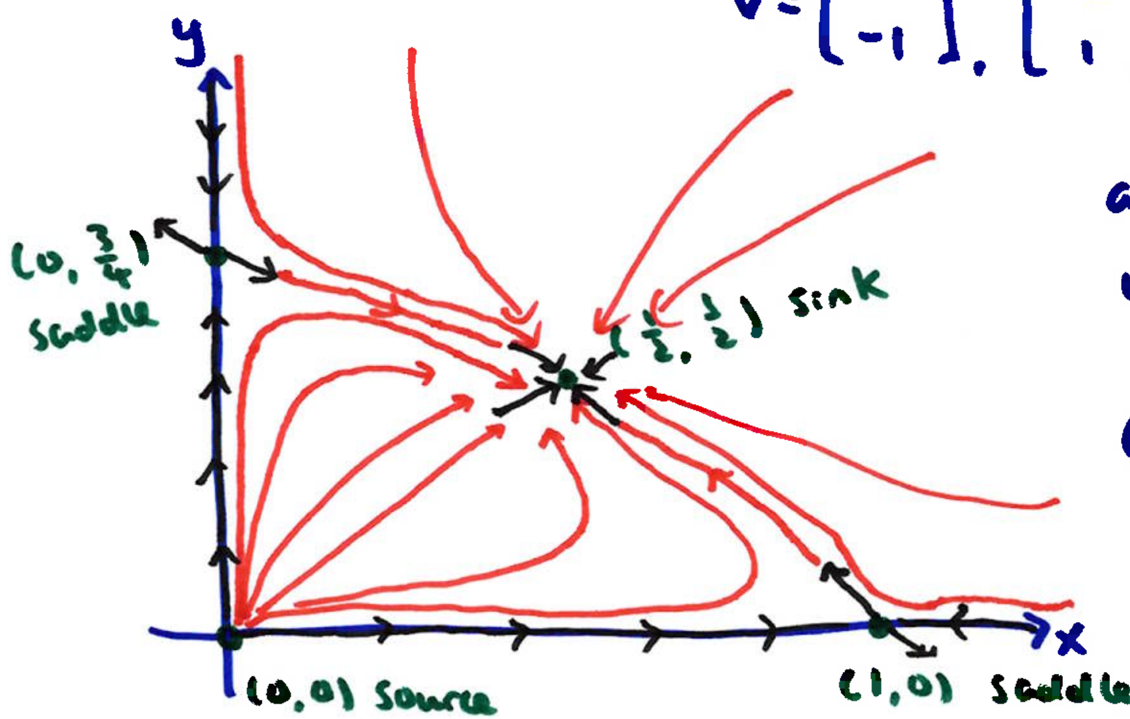
$$\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -5 \end{bmatrix}$$

$$J(0, \tfrac{3}{4}) = \begin{bmatrix} \frac{1}{4} & 0 \\ -\frac{3}{8} & -\frac{3}{4} \end{bmatrix} \quad \lambda = \tfrac{1}{4}, -\tfrac{3}{4} \qquad \text{saddle, unstable}$$

$$\vec{v} = \begin{bmatrix} 8 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$J(\tfrac{1}{2}, \tfrac{1}{2}) = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{4} & -\frac{1}{2} \end{bmatrix} \quad \lambda = -0.146, -0.854 \qquad \text{sink, asymp. stable}$$

$$\vec{v} = \begin{bmatrix} \sqrt{2} \\ -1 \end{bmatrix}, \begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix}$$



any $x(0) \neq 0$, $y(0) \neq 0$ will lead to coexistence ("weak" competition) example

$$x' = x(1-x-y)$$
$$y' = y\left(\frac{1}{2} - \frac{1}{4}y - \frac{3}{4}x\right)$$

cp: $(0,0)$, $(1,0)$, $(0,2)$, $\left(\frac{1}{2}, \frac{1}{2}\right)$

$$J(x,y) = \begin{bmatrix} 1-2x-y & -x \\ -\frac{3}{4}y & \frac{1}{2} - \frac{1}{2}y - \frac{3}{4}x \end{bmatrix}$$

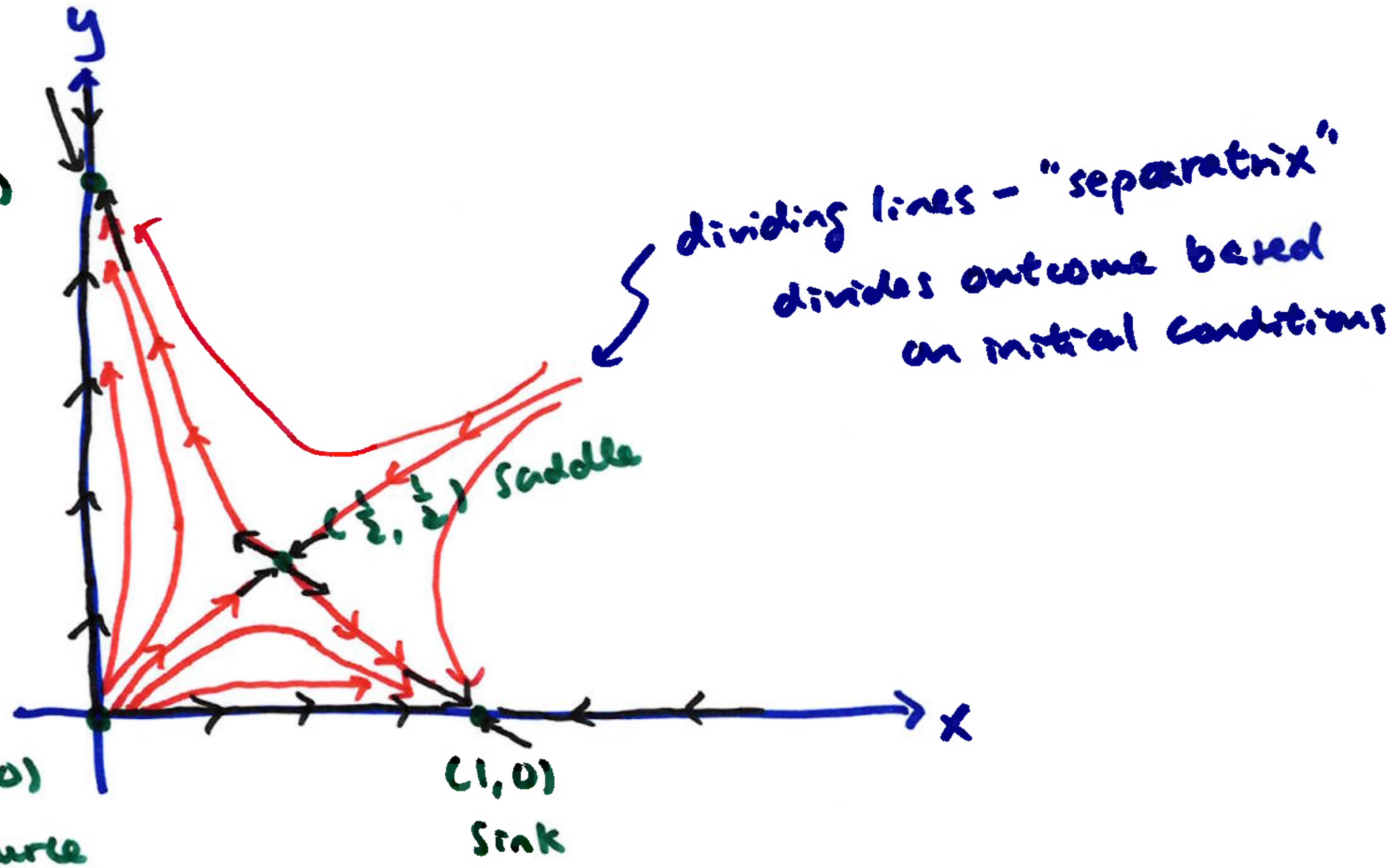
$$J(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \quad \lambda = 1, \frac{1}{2} \quad \vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{source}$$

$$J(1,0) = \begin{bmatrix} -1 & -1 \\ 0 & -\frac{1}{4} \end{bmatrix} \quad \lambda = -1, -\frac{1}{4} \quad \vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \end{bmatrix} \quad \text{sink}$$

$$J(0,2) = \begin{bmatrix} -1 & 0 \\ -\frac{3}{2} & -\frac{1}{2} \end{bmatrix} \quad \lambda = -1, -\frac{1}{2} \quad \vec{v} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{sink}$$

$$J\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -\frac{3}{8} & -\frac{1}{8} \end{bmatrix} \quad \lambda = 0.16, -0.78$$

$$\vec{v} = \begin{bmatrix} 1 \\ -1.32 \end{bmatrix}, \begin{bmatrix} 1 \\ 0.57 \end{bmatrix} \quad \text{saddle}$$



"strong" competition - one dies eventually

coexistence is unlikely

$$x' = a_1 x - b_1 x^2 - c_1 x y$$
$$y' = a_2 y - b_2 y^2 - c_2 x y$$
$$a_i, b_i, c_i > 0$$

$\boxed{\text{coexistence is likely if } b_1 b_2 > c_1 c_2}$

intrinsic limits   interaction

cooperation system: same as competition
except $c_i < 0$
(ants and aphids)

coop sys: coexistence very likely
but may lead to "doomsday" scenario
– both pop $\to \infty$